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A LOOSELY BERNOULLI COUNTEREXAMPLE MACHINE

BY

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ABSTRACT

In Rudolph's paper on minimal self joinings [7] he proves that a rank one mixing transformation constructed by Ornstein [5] can be used as the building block for many ergodic theoretical counterexamples. In this paper we show that Ornstein's transformation can be altered to create a general method for producing zero entropy, loosely Bernoulli counterexamples. This paper answers a question posed by Ornstein, Rudolph, and Weiss [6].

1. Introduction

The problem of finding a map T which is not Bernoulli, but the maps of the form $T \times T \times \cdots \times T$ are loosely Bernoulli, has been studied by Katok, Swanson, and Gerber. Katok created a simple weak mixing, but not mixing, map T such that $T \times T$ is loosely Bernoulli. Swanson extended this to a map T' such that all maps of the form $T' \times T' \times \cdots \times T'$ are loosely Bernoulli [8]. Gerber altered Katok's map to find a mixing transformation with all of the countable direct products of it with itself are loosely Bernoulli [2]. In this paper we exhibit a transformation, T, which is rank one and mixing. This transformation is an adaptation of one in [5]. Thus by King's theorem it has minimal self joinings and thus can serve as the basis for Rudolph's counterexample machine [4]. We also show it has $T \times \cdots \times T$ loosely Bernoulli, and all of the counterexamples T generates are loosely Bernoulli.

Ornstein's map is built by cutting and stacking $n-1$ blocks to form an n block. Two adjacent n blocks are separated by a "psuedorandom" number of spacer symbols. Most of our n block, the "mixing" portion, will be built this

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way. In section 3 we prove that if the mixing portion is most, rather than all, of the n block, T still has minimal self joinings.

On a small portion of the beginning of the n block we will arrange the $n-1$ blocks in a different manner. We will have many $n-1$ blocks with no spacer symbols between adjacent blocks, followed by many more $n-1$ blocks with 1 spacer symbol between adjacent $n-1$ blocks. We call these the "cyclic" regions of the n block. A point x in the first cyclic region has a name which for a long time is cyclic with period $h(n-1)$, the length of an $n-1$ block. Similarly a point y in the second cyclic region has a name under our partition which for a long time is cyclic with period $h(n-1)+1$. Thus the pair (x,y) has a name which for a long time is periodic with period $h(n-1)(h(n-1)+1)$. In section 4 we will use a nesting procedure to show these long periodic stretches in the (x,y) name make $T\times T$ loosely Bernoulli. As n increases we will increase the number of cyclic regions at the beginning of an n block. This will allow us to prove that $T\times\cdots\times T$ is loosely Bernoulli. In section 5 we show that a large class of transformations built from T are loosely Bernoulli.

2. Construction

The construction is by a cutting and stacking prcedure. Let the space $\Omega=(0,s)^{\mathbb{Z}}$. Our transformation T is the shift, $(T(\omega))_i=\omega i+1$. Let Q be the partition of Ω into two sets based on the value of ω_0. The zero block, $B(0)$, is the single symbol 0. The n block, $B(n)$, will be a string of $h(n)$ symbols of Q. To construct $B(n)$ inductively from $B(n-1)$ we need to choose $N(n)$, the number of $n-1$ blocks in the n block. We also need c_n, the number of cyclic regions at the beginning of an n block, and x_n, where $x_nN(n)\in\mathbb{Z}$ is the number of $n-1$ blocks in each cyclic region. We must have $c_nx_n\leq 1$. To specify the number of spacer s symbols between the $n-1$ blocks we need two sequences of integers. The first, $p_{1,n},p_{2,n},\ldots,p_{c_n,n}$, tells how many spacer symbols to put after an $n-1$ block in each of the cyclic regions. The second is the psuedorandom sequence, $a_{1,n},a_{2,n},\ldots,a_{(1-c_nx_n)N(n),n}$. We also need $S(n)$ so that all of the $a_{i,n}$ and $p_{i,n}$ are between 1 and $S(n)$. If these are all fixed then we construct the n block in the following way.

Let the jth cyclic region, $\overline{B_{j,n}}$, be defined as follows:

$$\overline{B_{j,n}}=(n-1\text{ block})\overbrace{sssss}^{p_{j,n}}(n-1\text{ block})\overbrace{sssss}^{p_{j,n}}\cdots(n-1\text{ block})\overbrace{sssss}^{p_{j,n}}.$$

Each cyclic region has $x_nN(n)$ $n-1$ blocks. Now we put these together to form

$$\overline{B_n}=\overline{B_{1,n}B_{2,n}}\cdots\overline{B_{c_n,n}},$$

the cyclic portion of the n block. To form the mixing portion, B_n^*, of the n block we define

$$B_{j,n}^* = \overbrace{ssssss}^{a_{i,n}}(n-1 \text{ block})\overbrace{ssssssss}^{S(n)-a_{i,n}}.$$

The mixing portion of an n block is created by putting those together,

$$B_n^* = B_{1,n}^* B_{2,n}^* \cdots B_{(1-c_n x_n N(n)),n}^*.$$

Then the n block is the cyclic portion followed by the mixing portion,

$$\begin{aligned} B(n) &= \overline{B_n} B_n^* \\ &= \overline{B_{1,n}} \cdots \overline{B_{c_n,n}} B_{1,n}^* B_{2,n}^* \cdots B_{(1-c_n x_n N(n)),n}^*. \end{aligned}$$

In order for names to have long enough cyclic regions to do the nesting procedures we choose $x_n \in \mathbb{Q}$ so that

$$x_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} (x_n)^p = \infty \quad \text{for all } p.$$

For technical reasons we choose $x_n = 0$ for all $n = 0, 2 \bmod 3$. Next we select $c_n \in \mathbb{Z}$ increasing to infinity but doing so slowly enough that $c_n x_n \to 0$.

We will use a sequence ϵ_n such that $\sum_N^\infty \epsilon_n < \epsilon_{N-1}$. Finally we choose $S(n+1), N(n), a_{i,j}$, and $p_{i,n}$ inductively. We choose $S(n+1)$, so large that

$$\frac{h(n-1)}{S(n+1)} < \epsilon_n.$$

Now we pick $N(n)$. Our first restriction on $N(n)$ is in order to make sure the measure space, Ω, is finite. We require that the number of spacers used in building the n block are a small fraction of the length of an n block. To do this, we need

$$\frac{S(n+1)}{h(n)} < \frac{S(n+1)}{N(n)h(n-1)} < \epsilon_n.$$

For any choice of the parameters listed above we will get a transformation. We want to make our choices so that the resulting transformation has a special mixing property for each n. First we need a few definitions.

We say that a point ω is in the n block if for some $i, i \le 0 \le i + h(n) - 1$, the symbols $Q(T^i(\omega)), \ldots, Q(T^{i+h(n)-1}(\omega))$ form the n block. We call that interval of integers $(i, i+h(n)-1)$ the n block around 0 for ω. We now generalize this to the case where we have a transformation $U = T^{l_1} \times \cdots \times T^{l_k}$ acting on Ω^k. We say that the k fold n overlap around 0 for $\omega_1, \ldots, \omega_k$, is the largest interval of integers $(-j, j')$ such that for all i, and $t \in (-j, j')$, $T^{tl_i}(\omega_i)$ is in the n block around 0. Similarly the mixing k fold n overlap around 0, $P_{0,n}(\omega_1, \ldots, \omega_k)$, is defined to be the largest interval of integers $(-j, j')$ such that for all i and $t \in (-j, j')$ $T^{tl_i}(\omega_i)$ is in the mixing portion of the n block.

We want to choose $N(n)$ large enough so that there exists a psuedorandom sequence, $a_{i,n}$, such that the n block has the following mixing property. For any k fold mixing overlap, (i, j), of n blocks, with $0 \leq k, |l_i| \leq n$, and for $1 \leq z_1, z_2, \ldots, z_k \leq h(n-2)$, one of three things must happen. Either

(1) the mixing overlap is extremely short ($< \epsilon_n h(n)$),

(2) there are points ω_t and $\omega_{t'}$, such that $l_t = l_{t'}$ and the n blocks for ω_t and $\omega_{t'}$ differ by less than $(h(n-1) + S(n))/l_t$, or

(3) $|\frac{1}{j-i-1}$(# of $t|T^{l_1 t}(\omega_1)$ is in the z_1th position of an $n-2$ block,
$T^{l_2 t}(\omega_2)$ is in the z_2th position of an $n-2$ block,...
$T^{l_k t}(\omega_k)$ is in the z_kth position of an $n-2$ block)
$-1/h(n-2)^n| < \epsilon_n$

LEMMA 1: (Rudolph) *There exists an N such that, for all $N(n) > N$, there exists a sequence $a_{i,n}, 1 \leq i \leq N(n)(1 - x_n c_n)$, which satisfies the above condition on mixing n overlaps.*

Proof: This is an application of the exponential rate of convergence for the weak law of large numbers and is proved by Rudolph [7]. ∎

We also need to have $x_n N(n) \in Z$, and if $x_n > 0$,

$$\frac{(2h(n-1))^{c_n}}{x_n N(n) h(n-1)} < \epsilon_n.$$

Choose $N(n)$ so that all of the above conditions are satisfied. Then select a sequence $a_{i,n}, 1 \leq i \leq N(n)(1 - x_n c_n)$, that satisfies our condition on mixing n overlaps. Now choose $p_{1,n+1}, \ldots, p_{c_{n+1},n+1}$ so that $h(n) + p_{1,n+1}, \ldots, h(n) + p_{c_{n+1},n+1}$ are relatively prime and

$$\frac{p_{j,n+1}}{h(n)} < \epsilon_n.$$

This can be done by a lemma in [2]. Proceeding in this manner we select $N(n), S(n), a_{i,n}$, and $p_{i,n}$.

3. T has minimal self joinings

In this section we show that our transformation T, even with the addition of cyclic portions in the n blocks, still has minimal self joinings. Therefore it can serve as the basis for Rudolph's counterexample machine.

Each point $(\omega, \omega') \in \Omega^2$ under $T \times T$ partitions $\mathbf{Z}$ into mixing n overlaps and gaps between the mixing n overlaps. $\overline{P_{0,n}}$ is the mixing n overlap containing 0, if it exists.

Definition 1: We say a mixing n overlap is **good** if it satisfies condition (3) above.

Definition 2: We say that an n block, $(i, i + h(n) - 1)$, for ω_1 and an n block, $(j, j + h(n) - 1)$, for ω_2 **line up** if $|j - i| \leq h(n-1) + S(n)$.

This next lemma is basically a restatement of condition (3) of our possibilities for mixing overlaps in the terms we will apply it.

LEMMA 2: *For any $(\omega, \omega') \in \Omega^2$, good $\overline{P}_{i,n}(\omega, \omega')$, and $A \times B, A$, and B are cylinder sets defining what happens from time 0 to time $h(n-3)$ ($A, B \in \bigvee_{-h(n-3)}^{0} T^i Q$), then*

$$\left|\frac{1}{|\overline{P}_{i,n}(\omega,\omega')|}(\#\text{ of } j \in \overline{P}_{i,n} | (T \times T)^j((\omega, \omega')) \in A \times B) - \mu(A)\mu(B)\right| < 10\epsilon_{n-3}.$$

Proof: Define $R(n)$ to be all points that are in an n block, but not in the last $h(n-1)$ levels of the n block. Thus

$$\mu(R(n)) > 1 - \frac{h(n-1)}{h(n)} - \sum_{i=n+1}^{\infty} \epsilon_i > 1 - \epsilon_{n-1}.$$

$$\frac{1}{|\overline{P}_{i,n}(\omega,\omega')|}(\#\text{ of } j \in \overline{P}_{i,n} | (T \times T)^j((\omega, \omega')) \in (A \times B) \cap (R(n-2) \times R(n-2)))$$

differs from

$$\frac{\mu \times \mu((A \times B) \cap (R(n-2) \times R(n-2))}{\mu \times \mu(R(n) \times R(n)))}$$

by less than $2\epsilon_{n-3}/(1 - 2\epsilon_{n-3})$ by Lemma 1. The latter differs from $\mu(A)\mu(B)$ by less than $2\epsilon_{n-3}/(1 - 2\epsilon_{n-3})$. As

$$\frac{1}{|\overline{P}_{i,n}(\omega,\omega')|}(\#\text{ of } j \in \overline{P}_{i,n} | (T \times T)^j(\omega, \omega') \in A \times B) =$$

$$\frac{1}{|\overline{P}_{i,n}(\omega,\omega')|}(\# \text{ of } j \in \overline{P}_{i,n} | (T\times T)^j(\omega,\omega') \in (A\times B)\cap(R(n-2)\times R(n-2)))$$
$$+\frac{1}{|\overline{P}_{i,n}(\omega,\omega')|}(\# \text{ of } j \in \overline{P}_{i,n} | (T\times T)^j(\omega,\omega') \in (A\times B)\cap(R(n-2)\times R(n-2))^C)$$

and the last term is less than $2\epsilon_{n-3}$, we get the desired result. ∎

THEOREM 1: *T is mixing.*

Proof: Let $A, B \in \bigvee_{-j}^{j} T^i(Q)$. Fix an $\epsilon > 0$. To compute $\mu(A\cap T^m(B))$ take a point ω generic for all cylinder sets. Choose n large enough so that $h(n-3) > 100j$, all but $\epsilon/100$ of the points in Ω are in the n block, $h(n-1)/h(n) < \epsilon/100$, and $\epsilon_{n-3} < \epsilon/10$. Choose $m = 2h(n-1)$.

Now $(\omega, T^{-m}(\omega)) \in A\times B$ implies $\omega \in A\cap T^m(B)$. None of the n blocks of ω and $T^{-m}(\omega)$ line up. Thus Lemma 2 applies to every mixing n overlap of $(\omega, T^{-m}\omega)$ and we have

$$\left|\mu(A)\mu(T^mB) - \frac{1}{|\overline{P}_{i,n}(\omega,T^{-m}\omega)|}(\# \text{ of } j\in\overline{P}_{i,n}| (T\times T)^j(\omega,T^{-m}\omega)\in A\times B)\right| < \frac{\epsilon}{10}.$$

Because ω is generic we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{0}^{n-1}\chi_{A\times B}(T\times T)^j(\omega,T^{-m}\omega) = \mu(A\cap T^m(B)).$$

As n overlaps make up all but $\epsilon/10$ of the times

$$|\mu(A\cap T^m(B)) - \mu(A)\mu(B)| < \epsilon. \quad ∎$$

Since T is rank one and mixing it is three mixing [3]. In [4] it is claimed that [3] can be extended to show that T is mixing of all orders. This could also be proven directly by an argument similar to the one in [7].

THEOREM: *T has minimal self joinings.*

Proof: T is mixing and by construction is rank one. Thus by King's structure theorem for rank one maps [4], T has minimal self joinings. ∎

4. $T \times \cdots \times T \cdots$ is loosely Bernoulli

In this section we will show that our addition of the "cyclic" part of the n block is enough to make $T \times \cdots \times T$ loosely Bernoulli. Without this cyclic part the direct product of Ornstein's transformation with itself is not loosely Bernoulli [6]. The proof builds on the fact that if x is a sequence with period n, y is a sequence with period m, and $\gcd(n, m) = 1$, then (x, y) has period nm. If $x, y \in \Omega$ and x is in the first cyclic section of an $n+1$ block, then the x name looks cyclic with period $h(n)$ in some region around 0. Likewise if y is in the second cyclic section of an $n+1$ block then the y name looks cyclic with period $h(n) + a_{2,n}$. Thus if the cyclic parts are long enough (x, y) looks cyclic with period $h(n)(h(n) + a_{2,n})$. Now if x' and y' are in similar positions, then in some region (x', y') is also cyclic with the same period. If the overlaps of the cyclic parts are long enough this implies we have a good $\overline{f}$ matching of the (x, y) name with the (x', y') name around 0. We will combine matching of this sort from different blocks to obtain our $\overline{f}$ matching.

The strategy for matching names is a nesting procedure of the matching outlined above. Assume that we already have a large set with a good matching on $3n-3$ overlaps. In the $3n-2$ block we introduce a cyclic part. This gives us a small set such that any two points in that set are $\overline{f}$ very close for a very long time. By Lemma 1 most long enough $3n$ overlaps have a high density of times where both points are in this set. First we find the times where both points are in the good set and apply our good long matching. Then we go through the unmatched regions and apply the shorter matching which we had on the $3n-3$ overlaps. We then check that this scheme gives us sets approaching measure 1 on which the $\overline{f}$ distance between any two points in the set goes to zero.

Define $M_{3n+1,k}$ to be all $(x_1, \ldots, x_k)$ such that

$x_1 \in$ first half of the $\overline{B_{1,3n}}$ in a $3n+1$ block,
$x_2 \in$ first half of the $\overline{B_{2,3n}}$ in a $3n+1$ block,...
$x_k \in$ first half of the $\overline{B_{k,3n}}$ in a $3n+1$ block.

If $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in M_{3n+1,k}$, then

$$\overline{f}_{F(3n+1)}((x_1, \ldots, x_k), (y_1, \ldots, y_k)) \leq \epsilon_{3n+1}$$

where $F(3n+1) = \frac{1}{2} x_{3n+1} N(3n+1) h(3n)$. This is because there exists some i and i', $0 \leq i, i' \leq \prod_{j=1}^{k} (h(3n) + p_{j,3n+1})$, so

$$T^i x_1, T^i x_2, \ldots, T^i x_k \quad \text{and} \quad T^{i'} y_1, T^{i'} y_2, \ldots, T^{i'} y_k$$

are all in the first position of the $3n$ block. The equation is true because the next $\frac{1}{2}x_{3n+1}N(3n+1)h(3n)-\max(i,i')$ symbols must match exactly and discrepancies occur with density less than

$$\frac{(2h(3n))^{c_{3n+1}}}{x_{3n+1}N(3n+1)h(3n)} < \epsilon_{3n+1}.$$

Define $B_{3n,k}$ to be all points $x=(x_1,\dots x_k)$ in Ω^k such that no two $3n$ blocks for x_i and x_j line up and $(0,\epsilon_{3n}h(3n))$ is in a $3n$ block for all i. Thus

$$\mu(B_{3n,k}) > (1-2k^2\epsilon_{3n}) = 1-\delta_{3n}.$$

Let $s_0=1$ and

$$s_{3n} = \left(\frac{x_{3n-2}}{2}\right)^{2k}\epsilon_{3n-2} + \left(1-\left(\frac{x_{3n-2}}{2}\right)^{2k}\right)s_{3n-3} + 4\delta_{3n-3}.$$

Thus $s_n \to 0$ because $\sum \delta_n < \infty$, $\sum (x_n)^{2k} = \infty$ and $\epsilon_n \to 0$.

LEMMA 3: *For any* $p,q \in B_{3n,k}$, $\overline{f}_{\epsilon_{3n}h(3n)}(p,q) \le s_{3n}$.

Proof: We prove the lemma by induction. It is obvious for $n=0$. Given $p,q \in B_{3n,k}$ we find an $i, 0 \le i \le k^2h(3n-1)$, such that none of the $3n$ blocks around 0 for an p_j lines up with the $3n$ block for some T^iq_k. Thus for at least $(x_{3n-2}/2)^{2k}$ of the $j, 0 \le j \le \epsilon_{3n}h(3n)$, we have $U^j(p) \in M_{3n-2,k}$ and $U^{i+j}(q) \in M_{3n-2,k}$. Call this set of j, J. By the comment above we have an ϵ_{3n-2} good matching of the p name from j to $j+F(3n-2)$ with the q name on the interval from $j+i$ to $j+i+F(3n-2)$.

Now we can use Lemma 1 to conclude for at least $1-3\delta_{3n-3}$ of the $t \notin J$ we have $U^t(p), U^{t+i}(q) \in B_{3n-3,k}$. For those good t the inductive hypothesis gives us a matching of the p name from t to $t+\epsilon_{3n-3}h(3n-3)$ with the q name from $t+i$ to $t+i+\epsilon_{3n-3}h(3n-3)$.

Because $\epsilon_{3n-3}h(3n-3)$ is much smaller than $F(3n-2)$ we can cover at least $1-4\delta_{3n-3}$ of the remainder of the interval from 0 to $\epsilon_{3n}h(3n)$ disjointly with pairs of these shorter intervals. Thus we get

$$\begin{aligned}\overline{f}_{\epsilon_{3n}h(3n)}(p,q) &\le (\frac{x_{3n-2}}{2})^{2k}\epsilon_{3n-2} + (1-(\frac{x_{3n-2}}{2})^{2k})s_{3n-3} + 4\delta_{3n-3}\\ &\le s_{3n}. \quad \blacksquare\end{aligned}$$

THEOREM 3: $U = T\times\cdots\times T$ *is loosely Bernoulli.*

Proof: As $\mu(B_{3n,k}) \to 1$ and $s_{3n} \to 0$, the theorem follows from the lemma.

COROLLARY 1: *$T \times \cdots \times T \times \cdots$ is loosely Bernoulli.*

Proof: As the $\overline{f}$ limits of loosely Bernoulli transformations are loosely Bernoulli, we have that the infinite direct product $T \times \cdots \times T \times \cdots$ is loosely Bernoulli. ∎

5. Counterexamples are loosely Bernoulli

If π is a permutation of a countable set V with only cycles of finite length, then define

$$S_\pi^{l(v)}(\omega_1, \omega_2, \ldots, \omega_n, \ldots) = (T^{l(1)}\omega_{\pi(1)}, T^{l(2)}\omega_{\pi(2)}, \ldots T^{l(n)}\omega_{\pi(n)}, \ldots).$$

We will only consider maps with $\sum l(v) \neq 0$ for all cycles of π. Other π create transformations that are not ergodic. This is the class of maps which are used in [7] to generate all of the counterexamples.

The matching procedure used in this section will be a generalization of the one in the previous section. Let V' be a finite π invariant subset of V. Without loss of generality $V' = (1, \ldots, k)$. Given a map of the form $U = S_\pi^{l(v)}$ we can find an m such that $\pi^m|_{V'} = \text{id}$. Wait until $\max_{v \in V'} l(v) < n$ and $c_n \geq m, |V'|$. Now we go through the $\overline{f}$ matching as before with the following alterations.

Define $L_v = (\sum_{\text{cycle containing } v} l(t))m/(\text{length of the cycle})$) and then let $L = \max_{v \in V'} |L_v|$. Thus $U^m|_{V'} = S_{\text{id}}^{L_v}$.

Define our set of good matching, $M_{3n+1,k}$, so that $(x_1, \ldots, x_k) \in M_{3n+1,k}$ if

$x_1 \in$ first half of the $\overline{B_{1,3n}}$ in a $3n+1$ block if $L_1 > 0$,
$x_1 \in$ second half of the $\overline{B_{1,3n}}$ in a $3n+1$ block if $L_1 < 0$,
$x_k \in$ first half of the $\overline{B_{k,3n}}$ in a $3n+1$ block if $L_k > 0$,
$x_k \in$ second half of the $\overline{B_{k,3n}}$ in a $3n+1$ block if $L_k < 0$.

Because for $n > L$ we have that

$$L(i), h(3n) + p_{1,3n+1}, \ldots, h(3n) + p_{c_{3n+1},3n+1}$$

are relatively prime for each i, we get that if

$$(x_1, \ldots x_{k'}), (y_1, \ldots y_{k'}) \in M_{3n+1,k}$$

then for some

$$0 \leq i, i' \leq m \prod_{j \in k'} (h(3n) + p_{j,3n+1}),$$

$$U^i x_1, U^i x_2, \ldots, U^i x_k \quad \text{and} \quad U^{i'} y_1, U^{i'} y_2, \ldots, U^{i'} y_k$$

are all in the first position of an $3n$ block and $\pi^i = \pi^{i'} = \text{id}$. Now, out of the next

$$\frac{x_{3n+1}N(3n+1)h(3n)m}{2L} - \max(i,i')$$

symbols only the first $m(m-1)(3n+1)$ and the last $m(m-1)(3n+1)$ are not certain to match. Thus

$$\overline{f}_{F(3n+1)}((x_1,\dots x_k),(y_1,\dots y_k)) \leq \frac{\sup(i,i')+6m^2n}{F(3n+1)} \leq 4L\epsilon_n$$

where $F(3n+1) = x_{3n+1}N(3n+1)h(3n)m/2L$.

Define $B_{3n,k} = (x_1,\dots x_k) \in \Omega^k$ such that no two x_i and x_j have $3n$ blocks around 0 that line up and each point has $(-L\epsilon_{3n}h(3n), L\epsilon_{3n}h(3n))$ in an $3n$ block:

$$\mu_{3n}(B_{3n,k}) > (1-2Lk^2\epsilon_{3n}) = 1-\delta_{3n}.$$

Let

$$s_{3n} = \left(\frac{x_{3n-2}}{2}\right)^{2k} 4L\epsilon_{3n} + \left(1-\left(\frac{x_{3n-2}}{2}\right)^{2k}\right)s_{3n-3} + 4\delta_{3n-3}.$$

LEMMA 4: *For any $p,q \in B_{3n,k}$, $\overline{f}_{K(n)}(p,q) \leq s_{3n}$.*

Proof: Given $p,q \in B_{3n,k}$, find i, $0 \leq i \leq k^2h(3n-1)$ such that for at least $(x_{3n-2}/2)^k$ of the j, $0 \leq j \leq K(n)$, $U^j(p) \in M_{3n-2}$, and $U^{i+j}(q) \in M_{3n-2}$. Call this set of j, J. For all $j \in J$ we can match the p name on the interval $(j, j+F(3n))$ with the q name on the interval $(j+i, j+i+F(3n))$ to within $4L\epsilon_{3n-2}/m$ in $\overline{f}$.

Now most of the unmatched symbols are in B_{3n-3} so we can use our previous matching. For $1-3\delta_{3n-3}$ of the unmatched symbols we have that both $U^t(p), U^{i+t}(q) \in B_{3n-3,k}$. As t increases, match blocks of length $K(3n-3)$ starting at t and $i+t$ if $U^t(p), U^{i+t}(q) \in B_{3n-3}$ and no symbol in the block has been used in a previous matching. Because $\epsilon_{3n-3}h(n-3)$ is much shorter than $F(3n-2)$ at most $1-4\delta_{3n-3}$ of the interval is not covered by any matching. Thus we get

$$\begin{aligned}\overline{f}_{\epsilon_{3n}h(3n)}(p,q) &\leq \left(\frac{x_{3n-2}}{2}\right)^{2k} 4L\epsilon_{3n-2} + \left(1-\left(\frac{x_{3n-2}}{2}\right)^{2k}\right)s_{3n-3} + 4\delta_{3n-3} \\ &\leq s_{3n}.\end{aligned}$$

THEOREM 4: *$S_\pi^{l(v)}$ is loosely Bernoulli.*

Proof: As $\mu(B_{3n,k}) \to 1$ and $s_{3n} \to 0$ the theorem follows from the following lemma for the case of finite V. The transformations with infinite V are loosely Bernoulli because they are the $\overline{f}$ limit of loosely Bernoulli transformations. ∎

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